2.7 Resistivity of isomorphous alloys and Nordheim’s rule  What are the maximum atomic and weight percentages of Cu that can be added to Au without exceeding a resistivity that is twice that of pure gold? What are the maximum atomic and weight percentages of Au that can be added to pure Cu without exceeding twice the resistivity of pure copper? (Alloys are normally prepared by mixing the elements in weight.)

Solution
From combined Matthiessen and Nordheim rule,
\[ \rho_{\text{Alloy}} = \rho_{\text{Au}} + \rho_{\text{Cu}}, \]
with \( \rho_{\text{Cu}} = CX(1-X) \). In order to keep the resistivity of the alloy less than twice of pure gold, the resistivity of solute (Cu), should be less than resistivity of pure gold, i.e. \( \rho_t = CX(1-X) < \rho_{\text{Au}} \). From Table 2.3, Nordheim coefficient for Cu in Au at 20°C is \( C = 450 \text{ n\(\Omega\) m} \). Resistivity of Au at 20°C, using \( \alpha_0 = 1/251 \text{ K}^{-1} \), is
\[ \rho = \rho_o[1 + \alpha_o(T - T_0)] = (22.8 \text{ n\(\Omega\) m} )[1 + \frac{1}{251} \text{ K}^{-1}(293 \text{ K} - 273 \text{ K})] = 24.62 \text{ n\(\Omega\) m} \]
Therefore the condition for solute (Cu) atomic fraction is \( CX(1-X) < 24.62 \text{ n\(\Omega\) m} \).
\[ X(1-X) < (24.62 \text{ n\(\Omega\) m})/(450 \text{ n\(\Omega\) m}) = 0.0547 \]
\[ X^2 - X + 0.0547 > 0 \quad \text{[Typo corrected; calculated values unchanged (5 Nov 07)]} \]
Solving the above equation, we have \( X > 0.0581 \) or 5.81%. Therefore the atomic fraction of Cu should be less than 0.0581 in order to keep the overall resistivity of the alloy less than twice the resistivity of pure Au. The weight fraction for Cu for this atomic fraction can be calculated from
\[ W_{\text{Cu}} = \frac{XM_{\text{Cu}}}{XM_{\text{Cu}} + (1-X)M_{\text{Au}}} = \frac{(0.0581)(63.54 \text{ g mol}^{-1})}{(0.0581)(63.54 \text{ g mol}^{-1}) + (1-0.0581)(196.67 \text{ g mol}^{-1})} \]
\[ = 0.01956 \text{ or 1.956%}. \]
Now, we discuss the case of Au in Cu, i.e. Au as solute in Cu alloy. Resistivity of Cu at 0°C is 15.7 n\(\Omega\) m. Therefore the resistivity of Cu at 20°C is
\[ \rho = \rho_o[1 + \alpha_o(T - T_0)] = 15.7 \text{ n\(\Omega\) m}[1 + \frac{1}{232} \text{ K}^{-1}(293 \text{ K} - 273 \text{ K})] = 17.05 \text{ n\(\Omega\) m} \]
Therefore the condition for solute (Au) atomic fraction is \( CX(1-X) < 17.03 \text{ n\(\Omega\) m} \). Nordheim coefficient for Au in Cu at 20°C is, \( C = 5500 \text{ n\(\Omega\) m} \).
\[ X(1-X) < (17.03 \text{ n\(\Omega\) m})/(5500 \text{ n\(\Omega\) m}) = 3.10\times10^{-3}. \]
\[ X^2 - X - 3.10\times10^{-3} < 0 \]
Solving the inequality we get the condition, \( X < 3.106\times10^{-3} \), required to keep the resistivity of alloy less than twice of pure Cu. The weight fraction for Cu for this atomic fraction can be calculated from
\[ W_{\text{Au}} = \frac{XM_{\text{Au}}}{XM_{\text{Au}} + (1-X)M_{\text{Cu}}} = \frac{(3.106\times10^{-3})(196.67 \text{ g mol}^{-1})}{(3.106\times10^{-3})(196.67 \text{ g mol}^{-1}) + (1-3.106\times10^{-3})(63.54 \text{ g mol}^{-1})} \]
\[ = 9.55 \times10^{-3} \text{ or 0.955%}. \]

2.13 Mixture rules
A 70% Cu - 30% Zn brass electrical component has been made of powdered metal and contains 15 vol. % porosity. Assume that the pores are dispersed randomly. Given that the resistivity of 70% Cu-30% Zn brass is 62 nΩ m, calculate the effective resistivity of the brass component using the simple conductivity mixture rule, Equation 2.26 and the Reynolds and Hough rule.

Solution

The component has 15% air pores. Apply the empirical mixture rule in Equation 2.26. The fraction of volume with air pores is \( \chi = 0.15 \). Then,

\[
\rho_{\text{eff}} = \rho \frac{(1 + \frac{1}{2} \chi)}{(1 - \chi)} = 62 \text{ nΩ m} \left( \frac{1 + 0.5 \times 0.15}{1 - 0.15} \right) = 78.41 \text{ nΩ m}
\]

Reynolds and Hough rule is given by Equation 2.28 as

\[
\frac{\sigma - \sigma_{\text{alloy}}}{\sigma + 2\sigma_{\text{alloy}}} = \chi \frac{\sigma_{\text{air}} - \sigma_{\text{alloy}}}{\sigma_{\text{air}} + 2\sigma_{\text{alloy}}}
\]

For the given case \( \sigma_{\text{air}} = 0 \), \( \sigma_{\text{alloy}} = (62 \text{ nΩ m})^{-1} \). Substituting the conductivity values in the RHS of the equation we have

\[
\chi \frac{\sigma_{\text{air}} - \sigma_{\text{alloy}}}{\sigma_{\text{air}} + 2\sigma_{\text{alloy}}} = (0.15) \left( \frac{0 - (62 \text{ nΩ m})^{-1}}{0 + 2(62 \text{ nΩ m})^{-1}} \right) = -0.075 \text{ g g}
\]

Solving for effective conductivity, we have \( \sigma = 1.2753 \times 10^7 \text{ Ω}^{-1} \text{ m}^{-1} \)

\[
\therefore \rho_{\text{eff}} = 78.41 \times 10^{-9} \text{ Ω m} \text{ or } 78.41 \text{ nΩ m}.
\]

Hence the values obtained are the same. Equation 2.26 is in fact the simplified version of Reynolds and Hough rule for the case when the resistivity of dispersed phase is considerably larger than the continuous phase.

2.25 Skin effect

a. What is the skin depth for a copper wire carrying a current at 60 Hz? The resistivity of copper at 27 °C is 17 nΩ m. Its relative permeability is \( \mu_r \approx 1 \). Is there any sense in using a conductor for power transmission with a diameter of more than 2 cm?

b. What is the skin depth for an iron wire carrying a current at 60 Hz? The resistivity of iron at 27 °C is 97 nΩ m. Assume that its relative permeability is \( \mu_r \approx 700 \). How does this compare with the copper wire? Discuss why copper is preferred over iron for power transmission even though the iron is nearly 100 times cheaper than copper.

Solution

a. The conductivity is \( 1/\rho \). The relative permeability \( (\mu_r) \) for copper is 1, thus \( \mu_{\text{Cu}} = \mu_o \). The angular frequency is \( \omega = 2\pi f = 2\pi(60 \text{ Hz}) \). Using these values in the equation for skin depth (\( \delta \)):

\[
\delta = \frac{1}{\sqrt{\frac{1}{\omega_o} \frac{1}{\rho_{\text{Cu}}} \mu_{\text{Cu}}}} = \frac{1}{\sqrt{\frac{1}{2 \pi (60 \text{ s}^{-1})} \left( \frac{4\pi \times 10^{-7} \text{ H m}^{-1}}{17 \times 10^{-9} \text{ Ω m}} \right)}}
\]

\[
\therefore \delta = 0.00847 \text{ m or } 8.47 \text{ mm}
\]
This is the depth of current flow. If the radius of wire is 10 mm or more, no current flows through the core region and it is wasted. There is no point in using wire much thicker than a radius of 10 mm (diameter of 20 mm).

b. The conductivity is $1/\rho$. The relative permeability ($\mu_r$) for Iron is 700, thus $\mu_{Fe} = 700\mu_o$. The angular frequency is $\omega = 2\pi f = 2\pi(60\text{ Hz})$. Using these values in the equation for skin depth ($\delta$):

$$
\delta_{Fe} = \frac{1}{\sqrt{2\omega}\rho \mu_{Fe}} = \frac{1}{\sqrt{\frac{1}{2}\left(\frac{2\pi 60 \text{s}^{-1}}{700}\frac{4\pi \times 10^{-7} \text{ H m}^{-1}}{97 \times 10^{-9} \Omega \text{ m}}\right)}}
$$

:. $\delta = 0.000765 \text{ m or } 0.765 \text{ mm}$

Thus the skin depth is 0.765 mm, about 11 times less than that for copper.

To calculate the resistance we need the cross sectional area for conduction. The material cross sectional area is $\pi r^2$ where $r$ is the radius of the wire. But the current flow is within depth $\delta$. We deduct the area of the core, $\pi (\rho - \delta)^2$, from the overall area, $\pi r^2$, to obtain the cross sectional area for conduction.

### 2.26 Thin films

a. Consider a polycrystalline copper film that has $R = 0.40$. What is the approximate mean grain size $d$ in terms of the mean free path $\lambda$ in the bulk that would lead to the polycrystalline Cu film having a resistivity that is $1.5\rho_{bulk}$. If the mean free path in the crystal is about 40 nm at room temperature, what is $d$?

b. What is the thickness $D$ of a copper film in terms of $\lambda$ in which surface scattering increases the film resistivity to $1.2\rho_{bulk}$ if the specular scattering fraction $p$ is 0.5?

c. Consider the data of Lim et al. (2003) presented in Table 2.13. Show that the excess resistivity, i.e. resistivity above that of bulk Cu, is roughly proportional to the reciprocal film thickness.

| Table 2.13 Resistivity $\rho_{film}$ of a copper film as a function of thickness $D$. |
|---|---|---|---|---|---|---|---|---|
| $D$ (nm) | 8.6 | 17.2 | 34.4 | 51.9 | 69 | 85.8 | 102.6 | 120.3 | 173.2 | 224.3 |
| $\rho_{film}$ (n$\Omega$m$^2$) | 121.8 | 75.3 | 46.1 | 38.5 | 32.1 | 25.2 | 22.0 | 20.5 | 19.9 | 18.8 |

**Solution**

a. Mayadas-Shatkez formula estimates the equivalent resistivity of the resistivity of polycrystalline sample as

$$
\frac{\rho}{\rho_{crystal}} \approx 1 + 1.33\beta
$$

where,

$$
\beta = \frac{\lambda}{d \left(\frac{R}{1 - R}\right)}
$$

Using the probability of reflection at a grain boundary $R = 0.40$, in the above formula we have

$$
\frac{1.5\rho_{crystal}}{\rho_{crystal}} \approx 1 + 1.33\frac{\lambda}{d \left(\frac{0.4}{1 - 0.4}\right)}
$$
\[ 1.5 = 1 + 0.89 \frac{\lambda}{d} \quad \text{or} \quad 1.5 - 1 = 0.89 \frac{\lambda}{d} \]

\[ d = 1.77 \lambda \]

If the mean free path in the crystal is \( \lambda = 40 \) nm, then the mean grain size is \( d = 1.77(40 \text{ nm}) = 70.8 \text{ nm} \).

b. Surface scattering resistivity is given by Equation 2.60 as

\[ \frac{\rho}{\rho_{\text{bulk}}} \approx 1 + \frac{3\lambda}{8D} (1 - p) \quad \frac{D}{\lambda} > 0.3 \]

Using \( p = 0.5 \), we have

\[ 1.2 \frac{\rho_{\text{bulk}}}{\rho_{\text{bulk}}} \approx 1 + \frac{\lambda \cdot 3}{D \cdot 8} \cdot (0.5) \]

\[ 1.2 \approx 1 + 0.1875 \frac{\lambda}{D} \]

**Simplifying we have** \( D \approx 0.9375 \lambda \).